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STOKES APPROXIMATION FOR TWO-DIMENSIONAL INDUCTIONLESS MAGNETOHYDRODYNAMIC FLOWS*

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The applicability of the Stokes approximation for describing steady MHD flows in infinitely long cylinders when there is no motion along the generator is examined. The external magnetic field only has a rotational component; it is stationary, coplanar with the flow plane and transverse to the surface of the cylinder. Sufficient conditions are indicated connecting the Reynolds and Hartmann numbers for which the non-linear terms in the Navier-Stokes equations can be ignored. Efficient a priori limits are obtained on various norms of the absolute error of the flow velocity.

The problem of establishing and analysing the region of validity of the Stokes approximation for describing the flow of an electrically conducting fluid was discussed at the 6th Riga Conference on Magnetoghydrodynamics /1/ and still remains largely unsolved. The first results in this area were summarized in the second chapter of /2/. The basic results obtained in /3/ to a certain extent filled the gap regarding the applicability of the Stokes approximation to some plane and axisymmetric MHD flows. Similar results were obtained in /4/ for two-dimensional flows in bounded multiply connected regions.

1. Statement of the problem. We consider the flow of a viscous incompressible electrically conducting fluid in a bounded closed region Ω, whose boundary Γ is formed by a finite number of piecewise-smooth closed contours Γ_i $(i = 0, 1, \ldots, n)$. This region is the section of an infinitely long cylinder by a plane perpendicular to its generator. Outside the cylinder, the magnetic permeability of the medium equals that of vacuum. The surface of the cylinder is impervious to the fluid. The flow is excited by the motion given on Γ which velocity $\mathbf{v}_\Gamma$. There is no flow component in the direction of the generator. A magnetic field of induction $\mathbf{B}$ is superimposed on the fluid flow. The vector field $\mathbf{B}$ is assumed stationary and coplanar with the flow plane. The disturbance of the field by the fluid flow is negligibly small (the inductionless case), the field does not vanish on $\Omega \bigcup \Gamma$ and satisfies the conditions ($\boldsymbol{\nu}$ is the outer normal to Γ)

$$\int_{\Gamma_i} \mathbf{B}\cdot\boldsymbol{\nu}\, d\Gamma_i = 0, \qquad \int_{\Gamma_0} |\mathbf{B}\cdot\boldsymbol{\nu}|^2\, d\Gamma_0 \neq 0$$

Regarding the velocity field $\mathbf{v}_\tau = v_\tau \boldsymbol{\tau}$ ($\boldsymbol{\tau}$ is the unit vector tangent to Γ), we assume that it may be continued inside Ω as a twice differentiable field $\mathbf{v}_0$. The continuation technique will be described below. In dimensionless variables, the flow is described by the boundary-value problem /2/

$$\nabla \times (\nabla \times \mathbf{v}) + \nabla (P + {}^1/_2 \operatorname{Re} |\mathbf{v}|^2) = \operatorname{Re} \mathbf{v} \times (\nabla \times \mathbf{v}) - \operatorname{Ha}^2 (\varkappa e + \mathbf{v} \times \mathbf{B}) \times \mathbf{B}, \quad \nabla\cdot\mathbf{v} = 0;\ e = \text{const},\ \boldsymbol{\varkappa} = \boldsymbol{\nu}\times \boldsymbol{\tau} \tag{1.1}$$

$$\mathbf{v}\cdot\boldsymbol{\nu}|_\Gamma = 0, \quad \mathbf{v}\cdot\boldsymbol{\tau}|_\Gamma = v_\tau \tag{1.2}$$

where $\operatorname{Re} > 0$ is the Reynolds number and $\operatorname{Ha} > 1$ is the Hartmann number; the field $\mathbf{B}$ is henceforth assumed known.

The Stokes approximation to the solution of problem (1.1), (1.2) is the pair $(\mathbf{v}_s, P_s)$ that satisfies system (1.1) with $\operatorname{Re} = 0$ and boundary conditions (1.2). The Stokes approximation error is the pair $(\delta\mathbf{v} = \mathbf{v} - \mathbf{v}_s, \delta P = P - P_s)$. In what follows, we derive bounds in various norms on the vector field $\delta\mathbf{v}$.

2. Generalized solutions and energy balance equations. As usual, /2, 5/, $J(\Omega)$ is the space of infinitely differentiable solenoidal vectors with finite support in Ω and $H(\Omega)$ is the Hilbert space obtained by closing $J(\Omega)$ in the norm generated by the scalar product

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$$[\mathbf{a}_1, \mathbf{a}_2]_H = \langle \nabla \times \mathbf{a}_1, \nabla \times \mathbf{a}_2 \rangle$$

Here and below,

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \int_\Omega \mathbf{a}_1 \cdot \mathbf{a}_2 \, d\Omega, \quad \| \mathbf{a} \|_p = \left\{ \int_\Omega | \mathbf{a} |^p \, d\Omega \right\}^{1/p}$$

and for $p = 2$ we omit the norm index.

The generalized solution of problem (1.1), (1.2) is the vector field $\mathbf{v}(x)$, $x \in \Omega$, which for any $\mathbf{\Phi} \subset J(\Omega)$ satisfies the integral identity

$$\langle \nabla \times \mathbf{v}, \nabla \times \mathbf{\Phi} \rangle = \mathrm{Re} \langle \nabla \times \mathbf{v}, \mathbf{\Phi} \times \mathbf{v} \rangle + \mathrm{Ha}^2 \langle \varkappa e + \mathbf{v} \times \mathbf{B}, \mathbf{B} \times \mathbf{\Phi} \rangle \tag{2.1}$$

and such that $\mathbf{v} - \mathbf{v}_0 \in H(\Omega)$.

We similarly define the generalized solution of the boundary-value problem for the Stokes approximation. Bounds on the norms of the Stokes approximation error will be determined from the energy balance equation for $\delta \mathbf{v}$. The equation is obtained from (2.1) if we set $\mathbf{\Phi} = \delta \mathbf{v}$ and represent $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{v}_s + \delta \mathbf{v}$

$$\| \nabla \times \delta \mathbf{v} \|^2 + \mathrm{Ha}^2 \| \delta \mathbf{v} \times \mathbf{B} \|^2 = \mathrm{Re} (\langle \nabla \times \mathbf{v}_s, \delta \mathbf{v} \times \mathbf{v}_s \rangle + \langle \nabla \times \delta \mathbf{v}, \delta \mathbf{v} \times \mathbf{v}_s \rangle) \tag{2.2}$$

The expression $e \langle \varkappa, \mathbf{B} \times \delta \mathbf{v} \rangle$ vanishes, because by the zero boundary conditions for $\delta \mathbf{v}$ there exists a stream function $\delta \psi$ which is related to $\delta \mathbf{v}$ by the equality $\delta \mathbf{v} = - \varkappa (\nabla \delta \psi)$. Hence

$$\langle \varkappa, \mathbf{B} \times [\varkappa \times (\nabla \delta \psi)] \rangle = \sum_{i=0}^{n} \int_{\Gamma_i} \delta \psi (\Gamma_i) \mathbf{B} \cdot \mathbf{v} \, d\Gamma_i$$

Since $\delta \psi$ is constant on Γ_i (its value is a priori unknown), the last sum vanishes by the conditions on the field $\mathbf{B}$.

We see from (2.2) that in order to obtain bounds on the norms of $\delta \mathbf{v}$ we need to bound the terms on the right-hand side of (2.2) by the norms $\| \nabla \times \mathbf{v}_s \|$, $\| \mathbf{v}_s \times \mathbf{B} \|$, $\| \mathbf{v}_0 \|_p$, $\| \nabla \times \mathbf{v}_0 \|$.

These terms contain the field $\mathbf{v}_s$, whose norms also need to be bounded. To this end, as in (2.2), we write the energy balance equation for $\mathbf{u}_s$:

$$\| \nabla \times \mathbf{u}_s \|^2 + \mathrm{Ha}^2 \| \mathbf{u}_s \times \mathbf{B} \|^2 = - \langle \nabla \times \mathbf{u}_s, \nabla \times \mathbf{v}_0 \rangle - \mathrm{Ha}^2 \langle \mathbf{u}_s \times \mathbf{B}, \mathbf{v}_0 \times \mathbf{B} \rangle, \quad \mathbf{u}_s = \mathbf{v}_s - \mathbf{v}_0 \in H(\Omega) \tag{2.3}$$

3. ***The fundamental inequality.*** To estimate the terms on the right-hand side of (2.2), we define a curvilinear system of coordinates in Ω associated with the geometry of the field $\mathbf{B}$. The local basis of this system is formed by the unit vectors $\boldsymbol{\beta}_1 = | \mathbf{B} |^{-1} \mathbf{B}$ и $\boldsymbol{\beta}_2 \perp \boldsymbol{\beta}_1$. In this basis

$$\delta \mathbf{v} = \delta v_1 \boldsymbol{\beta}_1 + \delta v_2 \boldsymbol{\beta}_2, \quad \mathbf{u}_s = u_1 \boldsymbol{\beta}_1 + u_2 \boldsymbol{\beta}_2$$

Applying the Cauchy inequality to the terms on the right-hand side of (2.2), we obtain the bounds

$$| \langle \nabla \times \mathbf{v}_s, \delta \mathbf{v} \times \mathbf{v}_s \rangle | \leqslant N_s \| \nabla \times \mathbf{v}_s \|, \quad | \langle \nabla \times \delta \mathbf{v}, \delta \mathbf{v} \times \mathbf{v}_s \rangle | \leqslant N_s \| \nabla \times \delta \mathbf{v} \|; \quad N_s = \| \delta \mathbf{v} \times \mathbf{v}_s \| \tag{3.1}$$

where it remains to estimate N_s. Expanding $\delta \mathbf{v}$ and $\mathbf{u}_s$ in the basis $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$, we use the Hölder inequality and the multiplicative inequality /5/ to obtain

$$N_s \leqslant N_0 + \| (\delta v_1 u_2 - \delta v_2 u_1) \varkappa \| \leqslant C_p \| \mathbf{v}_0 \|_p \| \nabla \times \delta \mathbf{v} \| + C_{1q} (\| \delta v_1 \| \| \nabla u_2 \|^{2/q} \| u_2 \boldsymbol{\beta}_2 \|^{1-2/q} + \| \nabla u_1 \| \| \nabla \delta v_2 \|^{2/q} \| \delta v_2 \boldsymbol{\beta}_2 \|^{1-2/q})$$

$$N_0 = \| \delta \mathbf{v} \times \mathbf{v}_0 \|, \quad C_p = \sqrt{2} \left(\frac{p}{p-2} \right)^{2/p} d^{1-2/p}, \quad C_{1q} = q^{1-2/q} d^{2/q}$$

($2 < p < 4$, $4 < q$ and d is the diameter of Ω).

Substituting this bound into (3.1) and applying Young's inequality, we obtain the following bounds on the terms in the right-hand side of (2.2):

$$| \langle \nabla \times \mathbf{v}_s, \delta \mathbf{v} \times \mathbf{v}_s \rangle | \leqslant C_p \| \mathbf{v}_0 \|_p \| \nabla \times \mathbf{v}_s \| \| \nabla \times \delta \mathbf{v} \| + C_{1q} \{ R \| \nabla \times \mathbf{v}_s \| \| \nabla \times \delta \mathbf{v} \| + q^{-1} \| \nabla \times \mathbf{v}_s \| \| \nabla \times \mathbf{u}_s \| \times [2 \varepsilon_1^{q/2} \| \nabla \times \delta \mathbf{v} \| + (q-2) \varepsilon_1^{-q(q-2)} \| \delta v_2 \boldsymbol{\beta}_2 \|] \} \tag{3.2}$$

$$|\langle\nabla\times\delta\mathbf{v},\delta\mathbf{v}\times\mathbf{v}_s\rangle|\leqslant C_p\|\mathbf{v}_0\|_p\|\nabla\times\delta\mathbf{v}\|^2+C_{1q}\{R\|\nabla\times\delta\mathbf{v}\|^2+$$
$$^1/_2q^{-1}\|\nabla\times\mathbf{u}_s\|[(q+2)\varepsilon_2^{2q/(q+2)}\|\nabla\times\delta\mathbf{v}\|^2+(q-2)\varepsilon_2^{-2q/(q-2)}\|\delta v_2\beta_2\|^2]\}$$
$$R=\|\nabla\times\mathbf{u}_s\|^{2/q}\|u_2\beta_2\|^{1-2/q}$$

Here we have used the bounds

$$\|\nabla u_i\|\leqslant\|\nabla\times\mathbf{u}_s\|,\ \|\nabla\delta v_i\|\leqslant\|\nabla\times\delta\mathbf{v}\|$$

It remains to obtain bounds on $\|\nabla\times\mathbf{u}_s\|$ and $\|u_2\beta_2\|$. To this end, we use the energy balance Eq.(2.3). The terms on the right-hand side of (2.3) are bounded by the Cauchy inequality, and isolating complete squares we obtain

$$\|\nabla\times\mathbf{u}_s\|\leqslant\|\nabla\times\mathbf{v}_0\|+{}^1/_2\mu\,\mathrm{Ha}\|\mathbf{v}_0\|$$
$$\|u_2\beta_2\|\leqslant{}^1/_2\|\nabla\times\mathbf{v}_0\|+\mu\,\mathrm{Ha}\|\mathbf{v}_0\|)\mathrm{Ha}^{-1}$$
$$\mu=M/m,\quad 0<m=\inf_{\Omega\cup\Gamma}|\mathbf{B}|,\quad M=\sup_{\Omega\cup\Gamma}|\mathbf{B}|$$

Substituting these bounds into (3.2), (3.1), and (2.3) and grouping similar terms, we obtain the basic inequality

$$({}^3/_4-C_p\,\mathrm{Re}\,\|\mathbf{v}_0\|_p-4^{2/q-1}C_{1q}M_{21}\,\mathrm{Re}\,\mathrm{Ha}^{2/q-1})\|\nabla\times\delta\mathbf{v}\|^2+ \tag{3.3}$$
$$[(m\,\mathrm{Ha})^2-(q-2)C_{2q}(C_{1q}\,\mathrm{Re}\,M_{12})^{2q/(q-2)}]\|\delta v_2\beta_2\|^2\leqslant$$
$$\mathrm{Re}\,\{C_p\|\mathbf{v}_0\|_1M_{41}\|\nabla\times\delta\mathbf{v}\|+4^{(4-q)/2q}C_{1q}M_{31}{}^2\,\mathrm{Ha}^{2/q}\times$$
$$[(q+1)\mathrm{Ha}^{-1}\|\nabla\times\delta\mathbf{v}\|+(q/2-1)\|\delta v_2\beta_2\|]\}$$
$$M_{kl}=k\|\nabla\times\mathbf{v}_0\|+\mu l\|\mathbf{v}_0\|,\ C_{2q}=[4^{6-q}(q+2)^{q+2}q^{-2q}]^{1/(q+2)}$$

In (3.3),

$$\varepsilon_1{}^q=\mathrm{Ha}^{-2+4/q},\ \varepsilon_2=[4^{2/q}q^{-1}(q+2)C_{1q}\,\mathrm{Re}\,M_{21}]^{(q+2)/(2q)}$$

4. ***Constructing the vector field*** $\mathbf{v}_0$. We will construct the vector field using a modification of the Hopf technique /6/. In view of the imperviousness of the boundary, the field $\mathbf{v}_0$ can be represented in the form $\mathbf{v}_0=\nabla\times(\chi_\varepsilon(x)\psi_0(x)\varkappa)=\chi_\varepsilon(x)\nabla\psi_0'\times\varkappa+\psi_0\nabla\chi_\varepsilon(x)\times\varkappa,\quad x\in\Omega,$ $\varepsilon>0$. Here the function $\psi_0(x)$ is twice differentiable in Ω and satisfies the boundary conditions

$$\psi_0|_\Gamma=0,\ \mathbf{v}\cdot\nabla\psi_0|_\Gamma=-v_\tau$$

The function $\chi_\varepsilon(x)$ has the form /7/

$$\chi_\varepsilon(x)=\begin{cases}1, & 0\leqslant\rho(x)<\varepsilon/2\\ \frac{1}{2}+\frac{3}{4}\cos\frac{2\pi}{\varepsilon}\left[\rho(x)-\frac{\varepsilon}{2}\right]- & \\ \quad-\frac{1}{4}\cos^3\frac{2\pi}{\varepsilon}\left[\rho(x)-\frac{\varepsilon}{2}\right], & \varepsilon/2\leqslant\rho(x)<\varepsilon\\ 0, & \varepsilon\leqslant\rho(x)\end{cases}$$

where $\rho(x)$ is the distance from the point $x\in\Omega$ to the nearest boundary contour; ε does not exceed half the minimum distance between the contours Γ_i.

By the choice of $\varkappa_\varepsilon(x)$, the field $\mathbf{v}_0$ vanishes outside Ω_ε - a strip of width ε adjoining Γ. It is therefore sufficient to determine the specific form of the function $\psi_0(x)$ in this strip, continuing it arbitrarily outside the strip with the required smoothness. If Γ satisfies the Lyapunov condition, then in $\Omega_\varepsilon\cup\Gamma$ we obtain

$$\psi_0(x)=-v_\tau(\sigma_x)\rho(x),\ v_\tau\in C_2(\Gamma) \tag{4.1}$$

where σ_x is the value of the natural parameter of the point x_Γ, which is the point of the contour Γ closest to $x\in\Omega$.

Let

$$M_\nabla=\sup_{\Omega_\varepsilon\cup\Gamma}|\nabla\psi_0(x)|,\quad M_\Delta=\sup_{\Omega_\varepsilon\cup\Gamma}|\Delta\psi_0(x)|$$

L_Γ is the total length of the contours of Γ, k is the local curvature of the contours of Γ, $K=\sup_{\Gamma_{ij}}|k|$, Γ_{ij} is the j-th smooth piece of the contour Γ_i and $\varepsilon<\varepsilon_0=\min[p_m/3,(2K)^{-1}]$.

Simple but tedious algebra leads to the bounds

$$\| \mathbf{v}_0 \|_p \leqslant M_\nabla M_p \varepsilon^{1/p}, \ \| \nabla \times \mathbf{v}_0 \| \leqslant M^{(2)} \varepsilon^{-1/2} \tag{4.2}$$

$$M_p = M_\nabla \left\{1 + 3\pi^{1-1/4p} 2^{-2-1/p} \left[\left(\frac{2^{2p-1}-1}{2p+1}\right)^{1/2} + \frac{1}{4}\left(\frac{2^{2p+3}-1}{2p+3}\right)^{1/2}\right]^{1/p} \left[\frac{\Gamma(3p+1/2)}{\Gamma(3p+1)}\right]^{1/2p}\right\} L_\Gamma^{1/p}$$

$$M^{(2)} = (M_\nabla \varepsilon + 90 M_\nabla) L_\Gamma^{1/2}$$

If $\psi_0(x)$ is given by (4.1), then

$$M_\nabla = \left\{\max_\Gamma v_\tau^2 + 2\varepsilon^2 \max_\Gamma \left(\frac{dv_\tau}{d\sigma}\right)^2\right\}^{1/2} \tag{4.3}$$

$$M_\Delta = 2\left\{K \max_\Gamma |v_\tau| + 4\varepsilon^2 \max_\Gamma \left|\frac{dv_\tau}{d\sigma}\right| \max_\Gamma \left|\frac{dk}{d\sigma}\right| + 2\varepsilon \max_\Gamma \left|\frac{d^2 v_\tau}{d\sigma^2}\right|\right\}$$

The quantity ε is selected so as to ensure the least order of increase in Ha of the sum $\| \nabla \times \mathbf{v}_0 \| + \mathrm{Ha} \| \mathbf{v}_0 \|$; specifically $\varepsilon = \varepsilon_0 \mathrm{Ha}^{-1}$.

5. ***The conditions for the admissibility of the Stokes approximation. Error bounds.*** In what follows $q > 8$, $p = 2q/(q-4)$. Substituting the bounds (4.2) into (3.3), we obtain the inequality

$$\{3/4 - \mathrm{Ha}^{2/q-1/2} \mathrm{Re}\,(4^{2/q-1}\mu_{21} C_{1q} + C_p M_\nabla M_p \varepsilon_0^{1/q})\} \| \nabla \times \delta\mathbf{v} \|^2 + \mathrm{Ha}^2 \{m^2 - (q-2) C_{2q} (C_{1q}\mu_{12}\mathrm{Re})^{2q/(q-2)} \mathrm{Ha}^{-(q-4)/(q-2)}\} \| \delta v_2 \boldsymbol{\beta}_2 \|^2 \leqslant \tag{5.1}$$

$$\mathrm{Re}\,\mathrm{Ha}^{2/q} \{[C_p M_\nabla M_p \varepsilon_0^{1/p} + 4^{(4-q)/2q} (1+q) C_{1q} \mu_{41}^2] \| \nabla \times \delta\mathbf{v} \| + 4^{(4-q)/2q} (q/2 - 1) C_{1q} \mu_{31}^2 \mathrm{Ha} \| \delta v_2 \boldsymbol{\beta}_2 \|\},$$

$$\mu_{kl} = k M_\nabla M_2 \varepsilon_0^{1/2} + l M^{(2)} \varepsilon_0^{-1/2}$$

We see from (5.1) that the conditions for the Stokes approximation to be admissible are identical with the conditions for the coefficients of $\| \nabla \times \delta\mathbf{v} \|^2$ and $\| \delta v_2 \boldsymbol{\beta}_2 \|^2$ to be positive. They thus have the form

$$\mathrm{Ha} \geqslant \mathrm{Re}^{2q/(q-4)} \max \{(4^{2q-1/2}\mu_{21} C_{1q} + 2C_p M_\nabla M_p \varepsilon_0^{1/p})^{2q/(q-2)}, [4/3\, m^{-2} C_{2q} (q-2)^{q-2} (C_{1q} M_{12})^{2q}]^{1/(q-4)}\} \tag{5.2}$$

Assume that Ha satisfies condition (5.2). Then separating complete squares in (5.1), we obtain the error bounds

$$\| \nabla \times \delta\mathbf{v} \| \leqslant 2\,\mathrm{Re}\,\mathrm{Ha}^{2/q} \Sigma_{21}, \ \| \delta v_2 \boldsymbol{\beta}_2 \| \leqslant 2\,\mathrm{Re}\,\mathrm{Ha}^{2/q-1} \Sigma_{12} \tag{5.3}$$

$$\Sigma_{kl} = k [C_p M_\nabla M_p \varepsilon_0^{1/p} + 4^{(4-q)/2q} (1+q) C_{1q} \mu_{41}^2] + l 4^{(4-q)/2q} (q/2 - 1) C_{1q} \mu_{31}^2, \quad k, l = 1, 2$$

Using the bounds (5.3) and the conditions imposed on the field B, we can obtain bounds on the norms $\| \delta\mathbf{v} \|_2$ and $\| \delta\mathbf{v} \|_r$, $2 < r < \infty$. To this end, we need a bound on $\| \delta\psi\varkappa \|$. Clearly, $\| \delta\mathbf{v} \times \mathbf{B} \| = \| \nabla\delta\psi \cdot \mathbf{B} \|$. In the basis $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ we have $\mathbf{B} \cdot \nabla\delta\psi = |\mathbf{B}| (\nabla\delta\psi \cdot \boldsymbol{\beta}_1)$. Therefore $\| \nabla\delta\psi \cdot \boldsymbol{\beta}_1 \| \leqslant \mu \| \delta v_2 \boldsymbol{\beta}_2 \|$. Since $\delta\psi$ is defined, apart from an additive constant, we may assume that $\delta\psi |_{\Gamma_0} = 0$, $\delta\psi |_{\Gamma_i} = C_i$ $(i = 1, \ldots, n)$ where C_i is a constant. The stream function defined in the multiply connected region Ω can be continued to a function $\overline{\delta\psi}$ defined in a simply connected region Ω_0 by taking $\overline{\delta\psi} = C_i$ outside the contours Γ_i. By the conditions imposed on B, there are sections of the contour Γ_0 on which $\mathbf{v} \cdot \mathbf{B} \neq 0$. Repeating for each of these sections the arguments that lead to the derivation of the Friedrichs inequality /5/, we obtain the bound

$$\mu \| \delta v_2 \boldsymbol{\beta}_2 \| \geqslant \| \nabla \overline{\delta\psi} \cdot \boldsymbol{\beta}_1 \| \geqslant d^{-1} \| \overline{\delta\psi\varkappa} \| > d^{-1} \| \delta\psi\varkappa \| \tag{5.4}$$

The bound for $\| \delta\psi\varkappa \|$ now follows from (5.4) and (5.3). Hence we obtain the required bound

$$\| \delta\mathbf{v} \| = | \langle \delta\psi\varkappa, \nabla \times \delta\mathbf{v} \rangle |^{1/2} \leqslant \mu d \| \delta v_2 \boldsymbol{\beta}_2 \|^{1/2} \| \nabla \times \delta\mathbf{v} \|^{1/2} \tag{5.5}$$

From (5.5) and the multiplicative inequality, we obtain a bound for $\| \delta\mathbf{v} \|_r$, $2 < r < \infty$,

$$\| \delta \mathbf{v} \|_r \leqslant \max (2,\ r/2)^{1-2/r} \| \nabla \times \delta \mathbf{v} \|^{1-2/r} \| \delta v_2 \boldsymbol{\beta}_2 \|^{1/r} \times \\ [\| \delta v_2 \boldsymbol{\beta}_2 \|^{1/r} + (\mu d)^{2/r} \| \nabla \times \delta \mathbf{v} \|^{1/r}] \quad (5.6)$$

If we only need the dependence on the parameters Re and Ha and the form of the constant multipliers is immaterial, the bounds (5.3), (5.5) may be rewritten as

$$\| \nabla \times \delta \mathbf{v} \| \leqslant C_\nabla \operatorname{Re} \operatorname{Ha}^{\alpha},\ \| \delta v_2 \boldsymbol{\beta}_2 \| \leqslant C_B \operatorname{Re} \operatorname{Ha}^{-1+\alpha} \\ \| \delta \mathbf{v} \| \leqslant C_v \operatorname{Re} \operatorname{Ha}^{-1/r+\alpha/2},\ \alpha = 2/q \quad (5.7)$$

(C_∇, C_B, C_v are functions of v_τ, q and the geometry of Ω and Γ).

6. Discussion of the results. The bounds (5.7) were reported without proof in /7/. The example in /6/ shows that the first two bounds in (5.7), by the arbitrariness of $\alpha < 1/4$, are close to the limiting bounds (in the sense of the dependence on Ha) with $\alpha = 0$. The limiting bounds are unattainable in the framework of the methods applied above, which are based on the multiplicative inequality, because they correspond to $q = \infty$, when the multiplicative inequality is meaningless.

As we have noted before, the inequalities describing the region of admissible values of Re and Ha were obtained from the conditions for the expressions on the left-hand side of inquality (5.1) to be positive. We see from the construction of these expressions that when inequality (5.2) is satisfied, viscous and electromagnetic forces in system (1.1) predominate over convective forces.

The derivation of bounds of the type (5.7) in stronger norms is problematic, because already $[\delta \mathbf{v}]_H = \| \nabla \times \delta \mathbf{v} \|$ does not tend to zero as $\operatorname{Ha} \to \infty$. Yet we can expect that the corresponding relative error norms will tend to zero.

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